ISRAEL JOURNAL OF MATHEMATICS 155 (2006), 57–80

COMPLETE WEIGHTS AND v-PEAK POINTS OF SPACES OF WEIGHTED HOLOMORPHIC FUNCTIONS

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ABSTRACT

We examine the geometric theory of the weighted spaces of holomorphic functions on bounded open subsets of $\mathbf{C}^n$, $\mathcal{H}v(U)$ and $\mathcal{H}v_o(U)$, by finding a lower bound for the set of weak*-exposed and weak*-strongly exposed points of the unit ball of $\mathcal{H}v_o(U)'$ and give necessary and sufficient conditions for this set to be naturally homeomorphic to U. We apply these results to examine smoothness and strict convexity of $\mathcal{H}v_o(U)$ and $\mathcal{H}v(U)$. We also investigate whether $\mathcal{H}v_o(U)$ is a dual space.

1. Introduction

The Banach–Stone Theorem tells us that, given compact Hausdorff sets K and L, the Banach space $C(K)$ of all continuous functions on K is isometrically isomorphic to $C(L)$ if and only if K and L are homeomorphic. The key to this result is to show that there is a one-to-one correspondence between K and the extreme points of the unit ball of $C(K)'$. This approach has proved

* The second author was supported by MCYT and FEDER Project BFM2002-01423.
Received March 31, 2005

useful in other function spaces; see [3], [17]. In [12] the authors commenced a geometric study of the weighted spaces of holomorphic functions $\mathcal{H}_{v_o}(U)$ and $\mathcal{H}_v(U)$ where U is a bounded open subset of $\mathbf{C}^n$ and v is a strictly positive continuous, weight on U which converges to 0 on the boundary of U. This paper may be considered as part of the isometric classification of weighted spaces of holomorphic functions. The isomorphic classification is considered in [11], [21], [22], [23], [24], [25], [26] and [27]. The phenomenon found in [3] and [17] appears in [12] again and we prove that an upper bound for the set of extreme points of the unit ball of $\mathcal{H}_{v_o}(U)'$ is $\{\lambda v(z)\delta_z : z \in U, \lambda \in \mathbf{C}, |\lambda| = 1\}$. We use $\mathcal{B}_v(U)$ to denote the set of all $z \in U$ such that $v(z)\delta_z$ is an extreme point of the unit ball of $\mathcal{H}_{v_o}(U)'$ and call it the v-boundary of U. In this paper we obtain a condition on the weight v that ensures that this upper bound is attained. As we shall see, when the set of extreme points of the unit ball of $\mathcal{H}_{v_o}(U)'$ is $\{\lambda v(z)\delta_z : z \in U, \lambda \in \mathbf{C}, |\lambda| = 1\}$, certain conclusions may be drawn concerning v, $\mathcal{H}_{v_o}(U)$ and $\mathcal{H}_v(U)$. In subsequent papers [13] and [14] we shall make use of the v-boundary to classify isometries between weighted spaces of holomorphic functions.

Let U be a bounded open subset of $\mathbf{C}^n$. A weight v on U is a bounded, strictly positive, continuous real valued function on U. We shall work with weights which converge to 0 on the boundary of U. By this we mean that given $\epsilon > 0$ there is a compact subset K of U such that $v(z) < \epsilon$ for z in $U \backslash K$. We will use $\mathcal{H}_v(U)$ to denote the space of all holomorphic functions f on U which have the property that $\|f\|_v := \sup_{z\in U} v(z)|f(z)| < \infty$ and endow $\mathcal{H}_v(U)$ with the norm $\|f\|_v$. Addition symmetry on v allows us to conclude addition properties of $\mathcal{H}_v(U)$. Such symmetry is provided by radial weights. A weight v on a bounded, balanced, open subset U of $\mathbf{C}^n$ is said to be radial if $v(\lambda z) = v(z)$ for all $z \in U$ and all λ in $\mathbf{C}$ with $|\lambda| = 1$. The space $\mathcal{H}_v(U)$ is a dual space and when v is radial it is even a bidual. Indeed, when v is radial it is the bidual of $\mathcal{H}_{v_o}(U)$, the set of all f in $\mathcal{H}_v(U)$ such that $|f(z)|v(z)$ converges to 0 as z converges to the boundary of U.

In this paper we give a sufficient condition for $\lambda v(z)\delta_z$ to be an exposed (or equivalently weak*-exposed) point of the unit ball of $\mathcal{H}_{v_o}(U)'$. (We do not assume that U is balanced or that v is radial.) A weight on the unit ball of $\mathbf{C}^n$ is said to be unitary if it is invariant under unitary matrices. We have a particular interest in determining under which conditions on v we have $\mathcal{B}_v(U) = U$. Such weights are said to be complete. We will show that a unitary weight on $B_{\mathbf{C}^n}$ is complete if $v\colon B_{\mathbf{C}^n} \to \mathbf{R}$ is a strictly decreasing unitary weight on the unit ball of $\mathbf{C}^n$ such that $v(x)$ is twice differentiable and

$$\Big(\frac{\partial}{\partial x_1}v(x)\Big)^2 - v(x)\Big(\frac{\partial^2}{\partial x_1^2}v(x)\Big) > 0$$

for all x of the form $(x_1, 0, \ldots, 0)$ with x_1 in $(0,1)$. We show that the norm at a point of the unit sphere of $\mathcal{H}_{v_o}(U)$ is Fréchet differentiable if and only if it is Gâteaux differentiable and give examples of both complete and non-complete weights on bounded open subsets of $\mathbf{C}^n$. In examining completeness of weights we shall see that it is $\log v$ that seems to give us more information rather than v itself. We prove that $\mathcal{B}_{v\times w}(U \times V) = \mathcal{B}_v(U) \times \mathcal{B}_w(V)$ and investigate the connection between the associated weights $\tilde{v}$ and $\tilde{v}_o$ and $\mathcal{B}_v(U)$.

The final section is concerned with applying our results on the v-boundary to obtain information on the geometry of $\mathcal{H}_v(U)$ and $\mathcal{H}_{v_o}(U)$ when the weight v converges to 0 on the boundary of U. We show: both of the spaces $\mathcal{H}_v(U)$ and $\mathcal{H}_{v_o}(U)$ are never smooth; when U is balanced, v is radial and each point of the v-boundary of U is a peak point, then neither $\mathcal{H}_v(U)$ nor $\mathcal{H}_{v_o}(U)$ is rotund.

ACKNOWLEDGEMENT: The authors would like to thank Professor K.D. Bierstedt for his comments and references.

During the summers of 2001, 2002 and 2003 the first author made short visits to the University of Valencia. He wishes to thank the Department of Mathematical Analysis there for its hospitality during those visits.

2. Structure of the v-boundary

In [12] we showed that the extreme points of the unit ball of $\mathcal{H}_{v_o}(U)'$ are contained in the set $\{\lambda v(z)\delta_z : z \in U, |\lambda| = 1\}$. It is readily seen that $v(z)\delta_z$ is an extreme point of $B_{\mathcal{H}_{v_o}(U)'}$ if and only if $\lambda v(z)\delta_z$ is an extreme point of $B_{\mathcal{H}_{v_o}(U)'}$ for any λ in $\mathbf{C}$ with $|\lambda| = 1$. Therefore, to specify the extreme points of $B_{\mathcal{H}_{v_o}(U)'}$ it is sufficient to give the set of all z for which $v(z)\delta_z$ is an extreme point of $B_{\mathcal{H}_{v_o}(U)'}$. This set we denote by $\mathcal{B}_v(U)$ and call the v-boundary of U. We showed in [12] that radial weights have radial v-boundaries in the sense that $z \in \mathcal{B}_v(U)$ if and only if $\lambda z \in \mathcal{B}_v(U)$ for all $\lambda \in \Gamma := \{\lambda \in \mathbf{C} : |\lambda| = 1\}$ and that the mapping μ: $U \to (\mathcal{H}_{v_o}(U)', \sigma(\mathcal{H}_{v_o}(U)', \mathcal{H}_{v_o}(U))$, $\mu(z) = v(z)\delta_z$, is a homeomorphism onto its image. From this it follows that $\mathcal{B}_v(U)$ is a G_δ subset of U.

Definition 1: Let U be a bounded open subset of $\mathbf{C}^n$. Given a subset A of U we denote by $\widehat{A}_{\mathcal{H}^\infty(U)}$ the closed convex $\mathcal{H}^\infty$ hull of A in U. That is

$$\widehat{A}_{\mathcal{H}^\infty(U)} := \{z \in U : |f(z)| \le \|f\|_A \text{ for all } f \in \mathcal{H}^\infty(U)\}.$$

Given A as above we shall use $\mathrm{cx}(A)$ to denote the closed convex hull of A in U.

THEOREM 2: *Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U which converges to 0 on the boundary of U, then $\widehat{\mathcal{B}_v(U)}_{\mathcal{H}^\infty(U)} = U$. In particular, if U is convex then $cx(\mathcal{B}_v(U)) = U$*

Proof: We may suppose without loss of generality that $v(z) \leq 1$ for all z in U. Suppose that the result is not true. Consider z in $U \setminus \widehat{\mathcal{B}_v(U)}_{\mathcal{H}^\infty(U)}$. By definition of $\widehat{\mathcal{B}_v(U)}_{\mathcal{H}^\infty(U)}$ we can find f_o in $\mathcal{H}^\infty(U)$ so that $f_o(z) > 1$ and $f_o(y) < 1$ for all y in $\mathcal{B}_v(U)$. Since f_o is bounded on U, f_o^n belongs to $\mathcal{H}_{v_o}(U)$ for any $n > 0$. By the Choquet Type Theorem [12, Theorem 18] we can find a $\mathbf{C}$-valued measure, ν_z, of bounded variation with support contained in $\overline{\mathcal{B}_v(U)}^U$ so that

$$(*) \qquad f(z) = \int_{\overline{\mathcal{B}_v(U)}^U} f(w)\, d\nu_z(w)$$

for all f in $\mathcal{H}_{v_o}(U)$. Applying $(*)$ to the functions f_o^n where $n > 0$ we get

$$f_o^n(z) = \int_{\overline{\mathcal{B}_v(U)}^U} f_o^n(w)\, d\nu_z(w)$$

for all $n > 0$. From this it follows that

$$|f_o(z)|^n \leq \|f_o\|^n_{\mathcal{B}_v(U)} \|\nu_z\|$$

for all $n > 0$ where $\|\nu_z\|$ is the total variation of ν_z. However, for n large enough we have $\|f_o\|^n_{\mathcal{B}_v(U)} \|\nu_z\| < 1$ while $|f_o(z)|^n > 1$. This contradiction implies that $\widehat{\mathcal{B}_v(U)}_{\mathcal{H}^\infty(U)} = U$. ∎

We shall use Δ to denote the open unit disc in the complex plane.

COROLLARY 3: *Let v be a continuous strictly positive radial weight on Δ which converges to 0 on the boundary of Δ. Then every point of the boundary of Δ is an accumulation point of $\mathcal{B}_v(\Delta)$.*

Proof: Suppose that there is z_o in the boundary of Δ which is not an accumulation point of $\mathcal{B}_v(\Delta)$. There we can find $\epsilon > 0$ so that $\mathcal{B}_v(\Delta) \cap B(z_o, \epsilon) = \emptyset$. It follows from [12, Lemma 5] that $\mathcal{B}_v(\Delta) \subset B(0, 1-\epsilon)$ contradicting Theorem 2. ∎

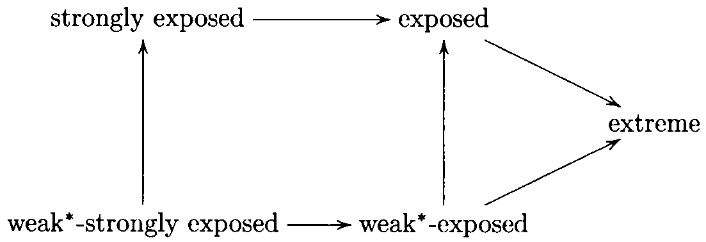
3. A Šmul'yan type Theorem

In order to obtain a lower bound for the v-boundary we will investigate the weak*-exposed points and weak*-strongly exposed points of the unit ball of $\mathcal{H}_{v_o}(U)'$. We recall the following definitions:

Definition 4: Let E be a complex Banach space. A point x in E is said to be an exposed point of the unit ball of E if there is $\phi \in E'$ of norm 1 such that $\mathbf{Re}(\phi(x)) = 1$ and $\mathbf{Re}(\phi(y)) < 1$ for all $y \in E$, $\|y\| \leq 1$, $y \neq x$. When $E = F'$ is a dual space and the vector ϕ which *exposes* x in B_E is in F, we shall say that x is *weak*-exposed* and that ϕ *weak* exposes* the unit ball of E at x.

Definition 5: A unit vector x in the Banach space E is *strongly exposed* if there is a unit vector $\phi \in E'$ so that $\phi(x) = 1$ and given any sequence $(x_k) \subseteq B_E$ with $\phi(x_k) \to 1$ we can conclude that x_k converges to x in norm. We will say that ϕ *strongly exposes* B_E at x. When $E = F'$ is a dual space and the vector ϕ which *strongly exposes* B_E is in F, we shall say that x is *weak*-strongly exposed* and that ϕ *weak*-strongly exposes* the unit ball of E at x.

The following diagram gives the relationship between the families of points we have introduced to date.



Each weak*-exposed point of the unit ball of E' is an extreme point of the unit ball of E'. Šmul'yan [34] shows that a point x in B_E weak*-exposes the unit ball of E' at ϕ if and only if the norm of E is Gâteaux differentiable at x with derivative $\mathrm{Re}(\phi)$ while in [35] he shows that a point x in B_E strongly weak*-exposes the unit ball of E' at ϕ if and only if the norm of E is Fréchet differentiable at x with derivative $\mathrm{Re}(\phi)$.

THEOREM 6: *Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U such that $v(z)$ tends to 0 as z converges to the boundary of U. Then for f_o in $B_{\mathcal{H}_{v_o}(U)}$, $\lambda \in \Gamma$ and $z \in U$ the following conditions are equivalent:*

G(a) *The norm on* $\mathcal{H}_{v_o}(U)$ *is Gâteaux differentiable at* f_o *with differential* $v(z)\mathbf{Re}(\lambda\delta_z)$.

G(b)

(i) $v(z)\lambda f_o(z) = 1$.

(ii) *If* $(z_k)_k$ *is a sequence in* U *and* $(\beta_k)_k$ *is a sequence of complex numbers of modulus 1 so that* $v(z_k)\beta_k f_o(z_k) \to 1$ *then* $(z_k)_k$ *and* $(\beta_k)_k$ *have subsequences* $(z_{k_i})_i$ *and* $(\beta_{k_i})_i$, *so that* $v(z_{k_i})\beta_{k_i}\delta_{z_{k_i}}$ *converges weak* to* $v(z)\lambda\delta_z$.

G(c) *The point* z *is the unique point in* U *with the property that* $v(z)\lambda f_o(z) = 1$.

F(a) *The norm on* $\mathcal{H}_{v_o}(U)$ *is Fréchet differentiable at* f_o *with differential* $v(z)\mathbf{Re}(\lambda\delta_z)$.

F(b)

(i) $v(z)\lambda f_o(z) = 1$.

(ii) *If* $(z_k)_k$ *is a sequence in* U *and* $(\beta_k)_k$ *is a sequence of complex numbers of modulus 1 so that* $v(z_k)\beta_k f_o(z_k) \to 1$ *then* $(z_k)_k$ *and* $(\beta_k)_k$ *have subsequences* $(z_{k_i})_i$ *and* $(\beta_{k_i})_i$, *so that* $v(z_{k_i})\beta_{k_i}\delta_{z_{k_i}}$ *converges in norm to* $v(z)\lambda\delta_z$.

F(c) f_o *strongly weak*-exposes the unit ball of* $\mathcal{H}_{v_o}(U)'$ *at* $v(z)\lambda\delta_z$.

Proof: By the Theorem of Šmul'yan in [34], G(a) implies $v(z)\delta_z$ is the unique point of the unit ball of $\mathcal{H}_{v_o}(U)'$ so that

$$\langle f_o, v(z)\delta_z\rangle = v(z)f_o(z) = 1.$$

In particular, z is the unique point in U so that $v(z)|f_o(z)| = 1$.

By the Theorem of Šmul'yan in [35] F(a) and F(c) are equivalent.

Suppose G(c) holds and that F(b) part (ii) is not true. Then we can find a neighbourhood V of 0 in $\mathcal{H}_{v_o}(U)'$, a sequence $(z_k)_{k=1}^{\infty}$ in U, a sequence $(\beta_k)_{k=1}^{\infty}$ of complex numbers with modulus 1 such that $v(z_k)\beta_k f_o(z_k) \to 1$ but none of the points $v(z_k)\beta_k\delta_{z_k}$ lies in $v(z)\lambda\delta_z + V$. We can choose a subsequence $(\beta_{k_i})_i$ of $(\beta_k)_k$ which converges to $\mu \in \Gamma$. We now choose a subsequence, (z_α), of (z_{k_i}) so that $(z_\alpha)_\alpha$ converges to some $\tilde{z}$ in $\overline{U}$. Since $f_o \in \mathcal{H}_{v_o}(U)$ and

$$\lim_{\alpha\to\infty} v(z_\alpha)\beta_\alpha f_o(z_\alpha) = 1$$

$\tilde{z}$ cannot belong to the boundary of U and hence is a point of U. As the function $z \to v(z)\delta_z$ is continuous we have that $\beta_\alpha v(z_\alpha)\delta_{z_\alpha}$ converges in norm to $\mu v(\tilde{z})\delta_{\tilde{z}}$. Therefore

$$(*) \qquad v(\tilde{z})\mu f_o(\tilde{z}) = \lim_{\alpha\to\infty} v(z_\alpha)\beta_\alpha f_o(z_\alpha) = 1.$$

By our assumption $\mu v(\tilde{z})\delta_{\tilde{z}}$ cannot be equal to $\lambda v(z)\delta_z$. Furthermore $z \neq \tilde{z}$, since the equality of z and $\tilde{z}$ along with $(*)$ will imply $\mu = \lambda$. From $(*)$ we now get that $v(\tilde{z})|f_o(\tilde{z})| = 1$ which contradicts G(c). Hence we see that G(c) implies F(b).

Now suppose that G(b) holds and that G(a) is not true. Then we can find f in $\mathcal{H}_{v_o}(U)$, $\epsilon > 0$ and a sequence $(\lambda_k)_k$ of real numbers converging to 0 so that

$$\|f_o + \lambda_k f\| - \|f_o\| - \lambda_k v(z)\mathbf{Re}(\lambda f(z)) \geq \epsilon|\lambda_k|$$

for every positive integer k. We choose, for every $k \in \mathbf{N}$, z_k, a point in U and β_k in $\mathbf{C}$ with $|\beta_k| = 1$ so that

$$v(z_k)\beta_k(f_o + \lambda_k f)(z_k) = \|f_o + \lambda_k f\|.$$

Then we have

$$\begin{aligned} 1 &= \|f_o\| \\ &\geq v(z_k)\mathbf{Re}(\beta_k f_o(z_k)) \\ &= v(z_k)\beta_k(f_o + \lambda_k f)(z_k) - \lambda_k v(z_k)\mathbf{Re}(\beta_k f(z_k)) \\ &\geq \|f_o + \lambda_k f\| - |\lambda_k|\|f\|. \end{aligned}$$

Since (λ_k) is a null sequence, $\|f_o + \lambda_k f\| - |\lambda_k|\|f\|$ converges to $\|f_o\|$ as k tends to ∞. Thus we have that $v(z_k)\mathbf{Re}(\beta_k f_o(z_k))$ and therefore $v(z_k)\beta_k f_o(z_k)$ converges to 1 as k tends to ∞. Hence, by part (ii) of G(b), there are subsequences $(z_{k_i})_i$ of $(z_k)_k$ and $(\beta_{k_i})_i$ of $(\beta_k)_k$ so that $v(z_{k_i})\beta_{\lambda_{k_i}}\delta_{z_{k_i}}$ converges weak* to $v(z)\lambda\delta_z$. We have

$$\|f_o + \lambda_{k_i} f\| \geq \mathbf{Re}(\lambda v(z)(f_o + \lambda_{k_i} f)(z)) = \|f_o\| + \lambda_{k_i} v(z)\mathbf{Re}(\lambda f(z)).$$

Therefore we have that

$$\begin{aligned} \epsilon|\lambda_{k_i}| &\leq |\|f_o + \lambda_{k_i} f\| - \|f_o\| - \lambda_{k_i} v(z)\mathbf{Re}(\lambda f(z))| \\ &= \|f_o + \lambda_{k_i} f\| - \|f_o\| - \lambda_{k_i} v(z)\mathbf{Re}(\lambda f(z)) \\ &= v(z_{k_i})\mathbf{Re}(\beta_{k_i}(f_o + \lambda_{k_i} f)(z_{k_i})) - \|f_o\| - \lambda_{k_i} v(z)(\mathbf{Re}\lambda f(z)) \\ &= v(z_{k_i})\mathbf{Re}(\beta_{k_i} f_o(z_{k_i})) + \lambda_{k_i} v(z_{k_i})\mathbf{Re}(\beta_{k_i} f(z_{k_i})) \\ &\qquad\qquad - \|f_o\| - \lambda_{k_i} v(z)\mathbf{Re}(\lambda f(z)) \\ &\leq |\lambda_{k_i}||v(z_{k_i})\mathbf{Re}(\beta_{k_i} f(z_{k_i})) - v(z)\mathbf{Re}(\lambda f(z))| \end{aligned}$$

for all k_i, which is impossible and shows that G(b) implies G(a).

Clearly we have that F(b) implies G(b).

Now suppose that F(b) holds and F(a) is false. Then we can find $\epsilon > 0$ and a sequence $(f_k)_k$ in $\mathcal{H}_{v_o}(U)$ converging to 0 so that

$$\|f_o + f_k\| - \|f_o\| - v(z)\mathbf{Re}(\lambda f_k(z)) \geq \epsilon \|f_k\|$$

for every positive integer k. We choose, for every $k \in \mathbf{N}$, z_k in U, and $|\beta_k| = 1$ so that

$$v(z_k)\beta_k(f_o + f_k)(z_k) > \|f_o + f_k\| - \frac{1}{k}\|f_k\|.$$

Thus we have

$$\begin{aligned} 1 &= \|f_o\| \\ &\geq v(z_k)\mathbf{Re}(\beta_k f_o(z_k)) \\ &= v(z_k)\beta_k(f_o + f_k)(z_k) - v(z_k)\mathbf{Re}(\beta_k f_k(z_k)) \\ &> \|f_o + f_k\| - \frac{1}{k}\|f_k\| - \|f_k\|. \end{aligned}$$

Since $\|f_o + f_k\| - \frac{1}{k}\|f_k\| - \|f_k\|$ converges to $\|f_o\|$ as k tends to ∞ we have that $v(z_k)\beta_k f_o(z_k) \to 1$. Part (ii) of F(b) now implies there are subsequences $(z_{k_i})_i$ of $(z_k)_k$ and $(\beta_{k_i})_i$ of $(\beta_k)_k$ so that $v(z_{k_i})\beta_{k_i}\delta_{z_{k_i}}$ converges in norm to $v(z)\lambda\delta_z$. Thus we have

$$\begin{aligned} \epsilon\|f_{k_i}\| \leq& \|f_o + f_{k_i}\| - \|f_o\| - v(z)\mathbf{Re}(\lambda f_{k_i}(z)) \\ <& v(z_{k_i})\beta_{k_i}(f_o + f_{k_i})(z_{k_i}) + \frac{1}{k_i}\|f_{k_i}\| - \|f_o\| - v(z)\mathbf{Re}(\lambda f_{k_i}(z)) \\ \leq& v(z_{k_i})\mathbf{Re}(\beta_{k_i} f_o(z_{k_i})) - v(z)\mathbf{Re}(\lambda f_o(z)) \\ &+ \|f_{k_i}\|\Big(\|v(z_{k_i})\mathbf{Re}(\beta_{k_i}\delta_{z_{k_i}}) - v(z)\mathbf{Re}(\lambda\delta_z)\| + \frac{1}{k_i}\Big) \\ \leq& \|f_{k_i}\|\Big(\|v(z_{k_i})\mathbf{Re}(\beta_{k_i}\delta_{z_{k_i}}) - v(z)\mathbf{Re}(\lambda\delta_z)\| + \frac{1}{k_i}\Big) \end{aligned}$$

and we arrive at a contradiction. Thus F(b) implies F(a). ∎

Thus we see that for spaces of the form $\mathcal{H}_{v_o}(U)$ Gâteaux and Fréchet differentiability of the norm at a given point coincide.

Definition 7: Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. We say that $z \in U$ is a v-peak point if there is $f \in \mathcal{H}_{v_o}(U)$ such that $v(z)f(z) = 1$ and $v(w)|f(w)| < 1$ for all $w \in U$ with $w \neq z$.

By Theorem 6, z is a v-peak point of U if and only if $v(z)\delta_z$ is a weak*-exposed point of $\mathcal{H}_{v_o}(U)'$.

PROPOSITION 8: *Let U be a bounded balanced open subset of $\mathbf{C}^n$ and v be a continuous strictly positive radial weight on U which converges to 0 on the boundary of U. If z is a v-peak point of U and $\lambda \in \Gamma$ then λz is also a v-peak point of U.*

Proof: If f weak*-exposes the unit ball of $\mathcal{H}_{v_o}(U)'$ at $v(z)\delta_z$ and $\lambda \in \Gamma$, define $f_\lambda \in \mathcal{H}_{v_o}(U)$ by $f_\lambda(z) = f(\overline{\lambda} z)$. Since v is a radial weight Theorem 6 will imply that f_λ weak*-exposes the unit ball of $\mathcal{H}_{v_o}(U)'$ at $v(\lambda z)\delta_{\lambda z}$. ∎

PROPOSITION 9: *Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. Then the set of v-peak points of U is dense in $\mathcal{B}_v(U)$.*

Proof: Since $\mathcal{H}_{v_o}(U)'$ is a separable dual space it has the Radon–Nikodým Property. Therefore the closed unit ball of $\mathcal{H}_{v_o}(U)'$ is the weak*-closed convex hull of

$$P = \{\lambda v(z)\delta_z : z \text{ is a } v\text{-peak point of } U, \lambda \in \Gamma\}.$$

It follows from [19, Theorem 13.B] that the set of extreme points of the unit ball of $B_{\mathcal{H}_{v_o}(U)'}$ is contained in the weak*-closure of P. By [12, Lemma 9] it follows that the set of v-peak points of U is dense in the v-boundary of U. ∎

4. Complete and incomplete weights

We now have a method of finding examples of complete weights on a bounded open set U. By Theorem 6 it is sufficient to show that given z in U there is f_o in the unit ball of $\mathcal{H}_{v_o}(U)$ with the property that z is the unique point in U with the property that $v(z)|f_o(z)| = 1$. Let us give a condition on v which will imply this.

Definition 10: A continuous strictly positive weight v on $B_{\mathbf{C}^n}$ is said to be unitary if $v(z) = v(Az)$ for every $n \times n$ unitary matrix A.

Hence v is unitary if and only if $v(z) = v(w)$ whenever $\|z\| = \|w\|$. If $n = 1$ the concept of a unitary weight coincides with the concept of a radial weight. For $n = 2$ the weight $v(z) = (1 - \|z\|^{1+\frac{2}{\pi}\tan^{-1}(|z_2|/|z_1|)})$ is a radial weight which is not unitary. It is readily shown that if v is unitary then $\mathcal{B}_v(B_{\mathbf{C}^n})$ is unitary in the sense that $z \in \mathcal{B}_v(B_{\mathbf{C}^n})$ if and only if $Az \in \mathcal{B}_v(B_{\mathbf{C}^n})$ for all unitary matrices A (see the proof of [12, Lemma 5]).

A weight v on a bounded balanced domain U in $\mathbf{C}^n$ is said to be strictly decreasing if for every point z_o in the boundary of U the function $(0, 1) \to \mathbf{R}$,

$\lambda \to v(\lambda z_o)$ is strictly decreasing. In particular, this means that, when it exists, the derivative of v is negative along each ray in U with centre 0.

PROPOSITION 11: *Let v: $B_{\mathbf{C}^n} \to \mathbf{R}$ be a continuous strictly positive strictly decreasing unitary weight on the unit ball of $\mathbf{C}^n$ which converges to 0 on the boundary of $B_{\mathbf{C}^n}$ such that $v(x)$ is twice differentiable and*

$$\Big(\frac{\partial v(x)}{\partial x_1}\Big)^2 - v(x)\frac{\partial^2 v(x)}{\partial x_1^2} > 0$$

for x of the form $(x_1, 0, \ldots, 0)$ with x_1 in $(0,1)$. Then the weak-exposed points of the unit ball of $\mathcal{H}_{v_o}(B_{\mathbf{C}^n})'$ is the set $\{v(z)\lambda\delta_z : \lambda \in \Gamma, z \in B_{\mathbf{C}^n}\}$.*

Proof: First observe that a suitably normalised constant function will weak*-expose the unit ball of $\mathcal{H}_{v_o}(B_{\mathbf{C}^n})'$ at $v(0)\delta_0$. Let us fix r. Consider z with $\|z\| = r$. We can find $r_1, \ldots, r_n \geq 0$ with $r_1^2 + \cdots + r_n^2 = r^2 < 1$ that allows us to write z as $z = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})$. Let $f(z) = e^{\alpha z_1}$ for $\alpha \in (0, \infty)$. Then

$$\begin{aligned} v(z)|f(z)| &= v(r)|e^{\alpha r_1 e^{i\theta_1}}| \\ &= v(r)e^{\alpha r_1 \cos\theta_1}. \end{aligned}$$

Hence the maximum of $v(r)|f(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})|$ subject to $r_1^2 + \cdots + r_n^2 = r^2$ occurs when $r_1 = r$ and $\theta_1 = 0$. Furthermore, we have that $v(r)|f(r, 0, \ldots, 0)|$ is strictly greater than $v(w)|f(w)|$ for any other w in $B_{\mathbf{C}^n}$ with $\|w\| = r$.

Let us now consider $v(z)f(z)$ restricted to the line segment

$$\{(x_1, 0, \ldots, 0) : x_1 \in (0,1)\}.$$

Differentiating $v(x_1, 0, \ldots, 0)e^{\alpha x_1}$ with respect to x_1 we get

$$\frac{\partial}{\partial x_1}(v(x_1, 0, \ldots, 0)e^{\alpha x_1}) = e^{\alpha x_1}\Big(\alpha v(x_1, 0, \ldots, 0) + \frac{\partial}{\partial x_1} v(x_1, 0, \ldots, 0)\Big).$$

Given $x_o \in (0,1)$, $v(x_1, 0, \ldots, 0)e^{\alpha_o x_1}$ will have a local extreme point at $(x_o, 0, \ldots, 0)$ when

$$\alpha_o = -\frac{1}{v(x_o, 0, \ldots, 0)}\frac{\partial v(x_o, 0, \ldots, 0)}{\partial x_1}.$$

As

$$\begin{aligned} &v(x_1, 0, \ldots, 0)^2 \frac{\partial}{\partial x_1}\Big(\frac{1}{v(x_1, 0, \ldots, 0)}\frac{\partial}{\partial x_1} v(x_1, 0, \ldots, 0)\Big) \\ &= v(x_1, 0, \ldots, 0)\frac{\partial^2}{\partial x_1^2} v(x_1, 0, \ldots, 0) - \Big(\frac{\partial}{\partial x_1} v(x_1, 0, \ldots, 0)\Big)^2 < 0, \end{aligned}$$

the function

$$\frac{1}{v(x_1,0,\ldots,0)}\frac{\partial v(x_1,0,\ldots,0)}{\partial x_1}$$

is strictly decreasing on $(0,1)$. Hence the point $(x_o,0,\ldots,0)$ is the only extreme point of $v(x_1,0,\ldots,0)e^{\alpha_o x_1}$ in $B_{\mathbf{C}^n}$ on the line segment $\{(x_1,0,\ldots,0) : x_1 \in (0,1)\}$. Differentiating $\frac{\partial}{\partial x_1}(v(x_1,0,\ldots,0)e^{\alpha x_1})$ with respect to x_1, we get that

$$\begin{aligned}\frac{\partial^2}{\partial x_1^2}(v(x_1,0,\ldots,0)e^{\alpha_o x_1}) =& e^{\alpha_o x_1}\Big(\alpha_0\Big(\alpha_o v(x_1,0,\ldots,0)+\frac{\partial}{\partial x_1}v(x_1,0,\ldots,0)\Big)\\ &+\alpha_o\frac{\partial}{\partial x_1}v\Big(x_1,0,\ldots,0+\frac{\partial^2}{\partial x_1^2}v(x_1,0,\ldots,0)\Big).\end{aligned}$$

As

$$\begin{aligned}\frac{\partial^2}{\partial x_1}(v(x_o,0,\ldots,0)e^{\alpha_o x_o}) =& e^{\alpha_o x_o}\Big(\alpha_o\Big(\alpha_o v(x_o,0,\ldots,0)+\frac{\partial}{\partial x_1}v(x_o,0,\ldots,0)\Big)\\ &+\alpha_o\frac{\partial}{\partial x_1}v(x_o,0,\ldots,0)+\frac{\partial^2}{\partial x_1^2}v(x_o,0,\ldots,0)\Big)\\ =& e^{\alpha_o x_o}\Big(-\Big(\frac{\partial}{\partial x_1^2}v(x_o,0,\ldots,0)\Big)^2/v(x_o,0,\ldots,0)\\ &+\frac{\partial^2}{\partial x_1^2}v(x_o,0,\ldots,0)\Big),\end{aligned}$$

it follows by assumption that

$$-\Big(\frac{\partial}{\partial x_1}v(x_o,0,\ldots,0)\Big)^2/v(x_o,0,\ldots,0)+\frac{\partial^2}{\partial x_1^2}v(x_o,0,\ldots,0)<0$$

and therefore the point $(x_o,0,\ldots,0)$ is the unique maximum of the function $v(x_1,0,\ldots,0)f(x_1,0,\ldots,0)$ for x_1 in $(0,1)$.

Now let w be any other point of $B_{\mathbf{C}^n}$. Let $r=\|w\|$. As the group of unitary matrices acts transitively on spheres we can find a biholomorphic mapping φ so that $\varphi((r,0,\ldots,0))=w$. As $v(w)=v((r,0,\ldots,0))$ it follows from Theorem 6 that $f\circ\varphi^{-1}$ weak*-exposed the unit ball of $\mathcal{H}(B_{\mathbf{C}^n})'$ at $v(w)\delta_w$. ∎

We note that the condition that

$$\Big(\frac{\partial v(x_1,0,\ldots,0)}{\partial x_1}\Big)^2-v(x_1,0,\ldots,0)\frac{\partial^2 v(x_1,0,\ldots,0)}{\partial x_1^2}>0$$

for x_1 in $(0,1)$ is equivalent to either of the two conditions listed below:

1. $\frac{1}{v(x_1,0,\ldots,0)}\frac{\partial v(x_1,0,\ldots,0)}{\partial x_1}$ is strictly decreasing on $(0,1)$.
2. The function $x_1\to\log v(x_1,0,\ldots,0)$ is concave down for x_1 in $(0,1)$.

As

$$\frac{\partial^2}{\partial x_1^2}\log(v_1(x_1,0,\ldots,0)v_2(x_1,0,\ldots,0)) = \frac{\partial^2}{\partial x_1^2}\log(v_1(x_1,0,\ldots,0)) + \frac{\partial^2}{\partial x_1^2}\log(v_2(x_1,0,\ldots,0))$$

and

$$\frac{\partial^2}{\partial x_1^2}\log(v(x_1,0,\ldots,0)^\alpha) = \alpha\frac{\partial^2}{\partial x_1^2}\log(v(x_1,0,\ldots,0)) \quad \text{for } \alpha > 0$$

we see that the condition in Proposition 11 is stable under finite products and positive powers.

COROLLARY 12: *Let $v\colon B_{\mathbf{C}^n} \to \mathbf{R}$ be a twice differentiable strictly decreasing strictly positive unitary weight which converges to 0 on the boundary of $B_{\mathbf{C}^n}$ such that $v(x_1,0,\ldots,0)$ is concave down for x_1 in $(0,1)$. Then the set of weak*-exposed points of the unit ball of $\mathcal{H}_{v_o}(B_{\mathbf{C}^n})'$ is $\{\lambda v(z)\delta_z : \lambda\in\Gamma, z\in B_{\mathbf{C}^n}\}$.*

Proof: Since $v(x_1,0,\ldots,0)$ is concave down $\frac{\partial^2}{\partial x_1^2}v(x_1,0,\ldots,0) < 0$ on $(0,1)$ and therefore

$$\Big(\frac{\partial}{\partial x_1}v(x_1,0,\ldots,0)\Big)^2 - v(x_1,0,\ldots,0)\frac{\partial^2}{\partial x_1^2}v(x_1,0,\ldots,0) > 0$$

for all $x_1 \in (0,1)$. ∎

Example 13: Given $\alpha > 0$, $\beta \geq 1$, each of the following weights on $B_{\mathbf{C}^n}$ is complete.

(a) $v_{\alpha,\beta}(z) = (1-\|z\|^\beta)^\alpha$.
(b) $w_{\alpha,\beta}(z) = e^{-\alpha/(1-\|z\|^\beta)}$.
(c) $v(z) = (\log(2-\|z\|))^\alpha$.
(d) $v(z) = (1-\log(1-\|z\|))^{-\alpha}$.
(e) $v(z) = \cos\left(\frac{\pi}{2}\|z\|\right)$.
(f) $v(z) = \cos^{-1}\|z\|$.
(g) *Finite products of the examples in (a) to (f).*

Proof: By the remark preceding Corollary 12 we may assume without loss of generality that $\alpha = 1$.

(a) Since

$$\frac{\partial}{\partial x_1}\Big(\frac{1}{v_{1,\beta}(x)}\frac{\partial v_{1,\beta}(x)}{\partial x_1}\Big) = -\beta\Big(\frac{(\beta-1)x^{\beta-2}+x^{2\beta-2}}{(1-x^\beta)^2}\Big)$$

this case follows from the Proposition 11.

(b) In this case we have that

$$\frac{\partial^2}{\partial x_1^2}\log w_{1,\beta} = \frac{-2\beta^2 x^{2(\beta-1)}}{(1-x^\beta)^3} - \frac{\beta^2 x^{\beta-2}}{(1-x^\beta)^2} + \frac{\beta x^{\beta-2}}{(1-x^\beta)^2}$$

and again this case follows from Proposition 11.

(c) We have that

$$\frac{\partial}{\partial x_1}\Big(\frac{1}{v(x)}\frac{\partial v(x)}{\partial x_1}\Big) = -\frac{\log(2-x)+1}{(2-x)^2(\log(2-x))^2}$$

and Proposition 11 gives us that v is complete.

(d) In this case

$$\frac{\partial^2}{\partial x_1^2}\log v(x) = -\frac{1}{(1-x)^2}\Big(\frac{1}{(1-\log(1-x))} - \frac{1}{(1-\log(1-x))^2}\Big).$$

Cases (e) and (f) follow immediately from the fact that $\cos(\frac{\pi}{2}x)$ and $\cos^{-1}(x)$ are concave down on the interval $(0,1)$.

Case (g) follows from the observation before Corollary 12. ∎

We note in passing that in finding examples of complete weights we have only used one family of exposing functions on $B_{\mathbf{C}^n}$, namely functions of the form $e^{\alpha z_1}$. Other families of exposing functions could well yield other examples of complete weights.

One might now conjecture that every continuous strictly positive weight which converges to 0 on the boundary of U is complete. The following example quickly dispels such illusions.

Let $0 < r < 1$ and let v be any strictly positive continuous weight on Δ with the property that $v(z) = c$ is constant for $|z| \leq r$ and $v(z) \to 0$ as z converges to the boundary of U.

Let i: $r\Delta \to \Delta$ be the natural inclusion $i(z) = z$, δ_Δ: $\Delta \to \mathcal{H}_v(\Delta)'$ be given by $\delta_\Delta(z) = \delta_z$. Then $\delta_\Delta \circ i$ is a bounded holomorphic mapping from $r\Delta$ into $\mathcal{H}_v(\Delta)'$ $(1 \geq \|v(z)\delta_\Delta \circ i(z)\| \geq \frac{1}{c}v(z) = 1$ for $\|z\| \leq r)$. Suppose U is an open subset of a Banach space E. Mujica [28] shows that there is a Banach space $G^\infty(U)$ and $\delta_U \in \mathcal{H}^\infty(U; G^\infty(U))$ such that the following universal property holds: given any Banach space F and any $f \in \mathcal{H}^\infty(U;F)$ there is a unique continuous linear operator T_f: $G^\infty(U) \to F$ such that $f = T_f \circ \delta_U$. Hence, by [28, Theorem 2.1] there is a continuous linear mapping $G^\infty(i)$: $G^\infty(r\Delta) \to \mathcal{H}_v(\Delta)'$ so that $\delta_\Delta \circ i = G^\infty(i) \circ \delta_{r\Delta}$. The map $G^\infty(i)$ is the transpose of restriction map

R from $\mathcal{H}_{v_o}(\Delta) \to \mathcal{H}^\infty(r\Delta)$, $f \to f|_{r\Delta}$. Since $\mathcal{H}_{v_o}(\Delta)$ contains all polynomials, it follows from [28, Lemma 5.1 (d), Proposition 4.9 (c) and Proposition 4.7 (b)] that R has $\sigma(\mathcal{H}^\infty(r\Delta), G^\infty(r\Delta))$-dense range. Applying [20, Corollary 2 to Proposition 3.12.2] we have that $G^\infty(i)$ is injective. By [2] we know that the unit ball of $G^\infty(r\Delta)$ has no extreme points. Hence, given $z \in r\Delta$ we can find ϕ_1, ϕ_2 in the unit ball of $G^\infty(r\Delta)$ with $\phi_1 \neq \phi_2$ such that $\delta_z = (\phi_1 + \phi_2)/2$. So $c\delta_z = (c\phi_1 + c\phi_2)/2$ and since $\|G^\infty(i)(c\phi)\| \leq \|\phi\|$, $G^\infty(i)(c\phi_1)$ and $G^\infty(i)(c\phi_2)$ are in the unit ball of $\mathcal{H}_v(\Delta)'$. As $G^\infty(i)$ is injective, $G^\infty(i)(c\phi_1) \neq G^\infty(i)(c\phi_2)$. Hence $c\delta_z = c\,\delta_\Delta \circ i(z) = c\,G^\infty(i) \circ \delta_{r\Delta}(z) = G^\infty(i)(c\delta_z)$ is the midpoint of the line segment from $G^\infty(i)(c\phi_1)$ to $G^\infty(i)(c\phi_2)$ and therefore is not an extreme point of the unit ball of $\mathcal{H}_v(\Delta)'$.

In order to examine the topological structure of the v-boundary of non-complete weights we consider the following example.

Example 14: Let $x_o \in (0,1)$ and consider a continuous strictly decreasing, strictly positive, radial weight w: $\{z : |z| \in [x_o, 1)\} \to \mathbf{R}$ which converges to 0 as $|z|$ tends to 1. Suppose that $w(x)$ is twice differentiable on $[x_o, 1)$ with $w'(x)^2 - w(x)w''(x) > 0$ for $x \in [x_o, 1)$ and $w'(x_o) < 0$. Define v: $\Delta \to \mathbf{R}^+$ by

$$v(z) = \begin{cases} w(x_o) & \text{if } |z| < x_o; \\ w(|z|) & \text{if } |z| \geq x_o. \end{cases}$$

Then $\mathcal{B}_v(\Delta) = \{z : |z| \in [x_o, 1)\}$.

Proof: Let

$$A = w''(x_o)/2, \quad B = w'(x_o) - w''(x_o)x_o$$

and

$$C = w(x_o) + (w''(x_o)/2)x_o^2 - w'(x_o)x_o.$$

Since $\lim_{x \to x_o} w'(x) < 0$, and $Ax^2 + Bx + C$ is continuous we can choose $\epsilon > 0$ so that $w''(x_o)(x - x_o) + w'(x_o) < 0$ and $Ax^2 + Bx + C > 0$ for $|x - x_o| < \epsilon$. Define a new weight u: $\Delta \to \mathbf{R}^+$ by

$$u(z) = \begin{cases} A(x_o - \epsilon)^2 + B(x_o - \epsilon) + C & \text{if } |z| \leq x_o - \epsilon; \\ A|z|^2 + B|z| + C & \text{if } x_o - \epsilon < |z| < x_o; \\ v(|z|) & \text{if } |z| \geq x_o. \end{cases}$$

Then u is a continuous strictly positive radial weight on Δ. Furthermore, for $|z| > x_o - \epsilon$, u is a strictly decreasing function of $|z|$. Consider the function $f_\alpha(z) = e^{\alpha z}$ for $\alpha > 0$. Let $x_1 \geq x_o$. Taking $\alpha = -u'(x_1)/u(x_1)$ the calculation of Proposition 11 shows that $u(z)|f_\alpha(z)|$ attains its maximum on $(x_o - \epsilon, 1)$ at

x_1. Furthermore, the value of $u(z)|f_\alpha(z)|$ at all other points of $(x_o - \epsilon, 1)$ is strictly less than the value at x_1. Since f_α is strictly increasing on $(0,1)$ and u is constant on $(0, x_o - \epsilon)$ we see that the value of $u(z)f_\alpha(z)$ at x_o is strictly greater than any other point of $(0, x_o - \epsilon)$. As $v(z) \le u(z)$ for $z \in \Delta$ and $v(x_1) = u(x_1)$ the function $v(z)|f_\alpha(z)|$ will also have a unique maximum at x_1. Hence $x_1 \in \mathcal{B}_v(\Delta)$. The fact that v and u are radial allows us to rotate any other point of $\{z : x_o \le |z| < 1\}$ onto a point with real coordinates and get that $\mathcal{B}_v(\Delta) = \{z : x_o \le |z| < 1\}$. ∎

This example shows that if v is a continuous strictly positive radial weight which converges to 0 on the boundary of U then the v-boundary need not be an open subset of U. We know from [12, Proposition 12] that it is always a G_δ set. We do not know if the v-boundary must always be a closed subset of U.

Unitary weights may be regarded as weights on the unit ball which exhibit a symmetry with respect to the set of unitary matrices. As we have seen, this symmetry allows us to give a sufficient condition for the weight to be complete. In the following example we give another sufficient condition for a weight to be complete. In this case our weights are *"rectangular"* weights on a square in the complex plane.

PROPOSITION 15: *Let v_1, v_2: $[0,1] \to \mathbf{R}$ be continuous strictly positive decreasing functions which satisfy $v_i'(x) < 0$ and $v_i'(x)^2 - v_i(x)v_i''(x) > 0$ for $x \in (0,1)$, and $v_i(1) = 0$ for $i = 1, 2$. Then the weight v: $(-1,1)\times(-1,1) \to \mathbf{R}^+$, $v(z) = v_1(|x|)v_2(|y|)$ is complete.*

Proof: First we observe that a suitable constant function weak*-exposes the unit ball of $\mathcal{H}_{v_o}((-1,1)\times(-1,1))'$ at $v(0)\delta_0$. Let us next consider the case when $x_o > 0$, $y_o > 0$. Let $\gamma = \alpha + i\beta$ and consider $v(z)|e^{\gamma z}|$. For $x, y > 0$ we have

$$v(z)|e^{\gamma z}| = v_1(x)v_2(y)e^{\alpha x}e^{-\beta y}.$$

Therefore

$$\begin{aligned}&\nabla(v_1(x)v_2(y)e^{\alpha x}e^{-\beta y})\\ &\quad = ((v_1'(x) + \alpha v_1(x))v_2(y)e^{\alpha x}e^{-\beta y}, v_1(x)(v_2'(y) - \beta v_2(y))e^{\alpha x}e^{-\beta y}).\end{aligned}$$

When $\alpha = -v_1'(x_o)/v_1(x_o)$ and $\beta = v_2'(y_o)/v_2(y_o)$ the point $x_o + iy_o$ is a critical point of $v(z)|e^{\gamma z}|$. Since $v_i'(x) < 0$ and $v_i'(x)^2 - v_i(x)v_i''(x) > 0$ for $x \in (0,1)$, $i = 1, 2$, this is the only critical point. Now

$$\frac{\partial^2}{\partial x \partial y}(v(z)|e^{\gamma z}|) = (v_1'(x) + \alpha v_1(x))(v_2'(y) - \beta v_2(y))e^{\alpha x}e^{-\beta y}$$

and we see that

$$\frac{\partial^2}{\partial x \partial y}(v(z)|e^{\gamma z}|)(x_o, y_o) = 0.$$

However,

$$\frac{\partial^2}{\partial x^2}(v(z)|e^{\gamma z}|)(x_o, y_o) = e^{\alpha x_o} e^{-\beta y_o} v_2(y_o)\Big(- \frac{v_1'(x_o)^2}{v_1(x_o)} + v_1''(x_o)\Big) < 0$$

and

$$\frac{\partial^2}{\partial y^2}(v(z)|e^{\gamma z}|)(x_o, y_o) = e^{\alpha x_o} e^{-\beta y_o} v_1(x_o)\Big(- \frac{v_2'(y_o)^2}{v_2(y_o)} + v_2''(y_o)\Big) < 0.$$

Hence

$$\frac{\partial^2}{\partial x \partial y}(v(z)|e^{\gamma z})|(x_o, y_o) = 0 < \Big(\frac{\partial^2}{\partial x^2}(v(z)|e^{\gamma z}|)(x_o, y_o)\Big)\Big(\frac{\partial^2}{\partial y^2}(v(z)|e^{\gamma z}|)(x_o, y_o)\Big).$$

As $\frac{\partial^2}{\partial x^2}(v(z)|e^{\gamma z}|)(x_o, y_o) < 0$ the Second Derivative Test implies that $v(z)|e^{\gamma z}|$ has a local maximum at $x_o + iy_o$. For $x < 0$ or $y < 0$ we note that $v_1(x) = v_1(-x)$ and $v_2(y) = v_2(-y)$. Hence $v_1(x)e^{\alpha x} < v_1(-x)e^{-\alpha x}$ or $v_2(y)e^{-\beta y} < v_2(-y)e^{\beta y}$. Therefore, the value $v(z)|e^{\gamma z}|$ at $x_o + iy_o$ is strictly greater than the values at all other points and so $x_o + iy_o \in \mathcal{B}_v((-1,1) \times (-1,1))$. For $x_o < 0$, $y_o > 0$ (resp. $(x_o < 0,\ y_o < 0)$, $(x_o > 0,\ y_o < 0)$) we replace $e^{\gamma z}$ by $e^{i\gamma z}$ (resp. $e^{-\gamma z}$, $e^{-i\gamma z}$).

Finally suppose $y_o = 0$ and $x_o > 0$. Let $\alpha = -v_1'(x_o)/v_1(x_o)$. It follows as in the case with unitary weights in Proposition 11 that $v_1(x_o)|e^{\alpha x_o}|$ is strictly greater than $v_1(x)|e^{\alpha x}|$ for $x > 0$, $x \neq x_o$. As $v_1(x)v_2(y)e^{\alpha x} < v(x_o)v_2(y_o)e^{\alpha x_o}$ for $y \neq 0$ it follows that $e^{\alpha z}$ weak*-exposes the unit ball of $\mathcal{H}_{v_o}((-1,1)\times(-1,1))'$ at x_o. An analogous argument shows that if $\beta = v_2'(y_o)/v_2(y_o)$ then $e^{i\beta z}$ weak*-exposes the unit ball of $\mathcal{H}_{v_o}((-1,1) \times (-1,1))'$ at iy_o. ∎

EXAMPLE 16: *The weight $v(z) = (1 - |x|^\alpha)(1 - |y|^\beta)$, $\alpha, \beta \geq 1$, satisfies the conditions of Proposition 15.*

5. The v-boundary of polydisc domains

Let us now examine how the v-boundary respects Cartesian products.

Definition 17: Let U be an open bounded subset of $\mathbf{C}^n$, v be a continuous strictly positive weight and F be a Banach space. We shall use $\mathcal{H}_v(U, F)$ to denote the Banach space of all holomorphic functions $f\colon U \to F$ such that $\|f\|_v := \sup_{z \in U} v(z)\|f(z)\| < \infty$.

Definition 18: Let U be an open bounded subset of $\mathbf{C}^n$, v be a continuous strictly positive weight and F be a locally convex space. We shall use $\mathcal{H}_{v_o}(U, F)$ to denote the space of all holomorphic functions $f: U \to F$ such that $v(z)f(z)$ converges to 0 as z converges to the boundary of U.

The following Proposition and Theorem are found in [4], [5] and [6]. Although they are stated there as giving isomorphisms between the spaces, an examination of the proofs shows that these isomorphisms are in fact isometries.

PROPOSITION 19 ([6, Corollary 30], [4, Satz 3.7]): *Let U be a bounded open subset of $\mathbf{C}^n$, v be a continuous strictly positive weight on U which converges to 0 on the boundary of U and F be a Banach space. Then $(\mathcal{H}_{v_o}(U; F), \|\cdot\|_v)$ is isometrically isomorphic to $(\mathcal{H}_{v_o}(U), \|\cdot\|_v)\epsilon F$.*

THEOREM 20 ([6, Corollary 42], [5, Satz 3.5]): *Suppose that U and V are bounded open subsets of $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively with v and w continuous strictly positive weights on U and V respectively each of which converges to 0 on the boundary of their respective domains. Then $\mathcal{H}_{(v\times w)_o}(U \times V)$ is isometrically isomorphic to $\mathcal{H}_{v_o}(U)\epsilon\mathcal{H}_{w_o}(V)$.*

PROPOSITION 21: *Let U be a bounded balanced open subset of $\mathbf{C}^n$ and v be a continuous strictly positive radial weight on U which converges to 0 on the boundary of U. Then $\mathcal{H}_{v_o}(U)'$ and $\mathcal{H}_{v_o}(U)$ have the metric approximation property.*

Proof: By [18, Examples III.1.4] $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$. (The proof in [18] is for the open unit disc Δ but is easily extended to arbitrary balanced domains in $\mathbf{C}^n$. Simply replace $\{z : \|z\| < r\}$ with rU.) Bierstedt, Bonet and Galbis, [7], have shown that $\mathcal{H}_{v_o}(U)$ has the metric approximation property. [18, Proposition III.2.5] implies that $\mathcal{H}_{v_o}(U)'$ has the metric approximation property. ∎

COROLLARY 22: *Let U and V be bounded balanced open subsets of $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively with v and w continuous strictly positive radial weights on U and V respectively each of which converges to 0 on the boundary of their respective domains. Then $\mathcal{H}_{(v\times w)_o}(U\times V)'$ is isometrically isomorphic to $\mathcal{H}_{v_o}(U)'\widehat{\otimes}_\pi\mathcal{H}_{w_o}(V)'$.*

Proof: Since $\mathcal{H}_{v_o}(U)'$ has the Radon–Nikodým property the result follows from the Proposition 21, [16, Theorem VIII.2.5], [16, Theorem VIII.4.6] and [16, Theorem VIII.4.7]. ∎

Taking strong duals again and using Proposition 21 we get that $\mathcal{H}_{v\times w}(U\times V)$ is isometrically isomorphic to $\mathcal{H}_v(U;\mathcal{H}_w(V))$. The fact that $\mathcal{H}_{v\times w}(U\times V)$ is isomorphic to $\mathcal{H}_v(U;\mathcal{H}_w(V))$ was proved in [9, Theorem 7] without any constraints on U, V, v or w. However, examining the proofs in [9] it is clear that this isomorphism is also an isometry.

PROPOSITION 23: *Let U and V be bounded open subsets of $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively with v and w continuous strictly positive radial weights on U and V respectively each of which converges to 0 on the boundary of their respective domains. Then $\mathcal{B}_{v\times w}(U\times V)=\mathcal{B}_v(U)\times\mathcal{B}_w(V)$. In particular, $v\times w$ is complete if and only if both v and w are complete.*

Proof: Apply Theorem 20 and [32, Theorem 1.3]. ∎

6. Associated weights

To construct examples of strictly decreasing weights v which converge to 0 on the boundary of U and which are non-complete, we introduce the associated weights $\tilde{v}_o$ and $\tilde{v}$. The concept of an associated weight has been considered by many authors (see [1], [8] and [33]).

Let U be an open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight which converges to 0 on the boundary of U. We define $w\colon U\to\mathbf{R}$ by $w(z)=1/v(z)$. The unit ball of $\mathcal{H}_{v_o}(U)$ is $\{f\in\mathcal{H}_{v_o}(U):|f(z)|\le w(z),\text{ for all } z\in U\}$ whereas the unit ball of $\mathcal{H}_v(U)$ is $\{f\in\mathcal{H}_v(U):|f(z)|\le w(z),\text{ for all } z\in U\}$. We define $\tilde{w}_o\colon U\to\mathbf{R}$ by

$$\tilde{w}_o(z)=\sup\{|f(z)|:f\in B_{\mathcal{H}_{v_o}(U)}\}$$

and $\tilde{w}\colon U\to\mathbf{R}$ by

$$\tilde{w}(z)=\sup\{|f(z)|:f\in B_{\mathcal{H}_v(U)}\}.$$

Let $\tilde{v}_o(z)=1/\tilde{w}_o(z)$ and $\tilde{v}(z)=1/\tilde{w}(z)$. Then $\tilde{v}_o$ and $\tilde{v}$ are continuous strictly positive weights which converge to 0 on the boundary of U and which satisfy $0<v\le\tilde{v}\le\tilde{v}_o$. We note that $\tilde{v}$ and $\tilde{v}_o$ are radial whenever U is balanced and v is radial. Furthermore, Hadamard's Three Circles Theorem implies that $\log\tilde{w}_o$ and $\log\tilde{w}$ are convex functions of $\log|z|$. When v is a continuous decreasing radial weight on the unit disc, [30,Theorem 2.6.6] tells us that $\log\tilde{w}_o$ and $\log\tilde{w}$ are subharmonic.

PROPOSITION 24: *Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U which decreases to 0 on the boundary of U. If $z \in \mathcal{B}_v(U)$ then $v(z) = \tilde{v}_o(z)$.*

Proof: First we observe that by definition $\|\delta_z\|_{\mathcal{H}_{v_o}(U)} = 1/\tilde{v}_o(z)$. Therefore, if $v(z)\delta_z$ is an extreme point of the unit ball of $\mathcal{H}_{v_o}(U)'$ then it must have norm 1. Hence

$$1 = \|v(z)\delta_z\|_{\mathcal{H}_{v_o}(U)'} = v(z)\|\delta_z\|_{\mathcal{H}_{v_o}(U)'} = v(z)\frac{1}{\tilde{v}_o(z)}.$$

Therefore, if z belongs to $\mathcal{B}_v(U)$ we have $v(z) = \tilde{v}_o(z)$. ∎

It appears that for continuous strictly positive radial weights which converge to 0 on the boundary of Δ there is a very deep relationship between the convexity of $\log v$ and the completeness of v. If v is radial we have

$$\log v(r) \text{ is a concave function of } r$$
$$\Downarrow$$
$$v \text{ is complete}$$
$$\Downarrow$$
$$\log v(r) \text{ is a concave function of } \log r.$$

Let v be a decreasing continuous, strictly positive, radial weight on Δ which converges to 0 on the boundary of Δ. Suppose there are $r_1, r_2 \in (0,1)$ with v strictly decreasing from 0 to r_1 and from r_2 to 1 but v is constant on the interval (r_1, r_2). Then the Maximum Modulus Principle will mean there is no function f in the unit ball of $\mathcal{H}_{v_o}(\Delta)$ which has a unique maximum at any point z with $|z|$ in (r_1, r_2). Hence by Proposition 24 any z with $r_1 < |z| < r_2$ cannot be a v-peak of Δ. Therefore, by Proposition 9 any such z is not in $\mathcal{B}_v(U)$.

Proposition 24 has some important corollaries that are worth pointing out.

COROLLARY 25: *If v is a continuous strictly positive complete weight on the unit ball of $\mathbf{C}^n$ which converges to 0 on the boundary of $B_{\mathbf{C}^n}$, then $v = \tilde{v}$. In particular, if $v(x)$ is twice differentiable strictly decreasing unitary weight and $\log v(x, 0, \ldots, 0)$ is concave down for x in $(0,1)$, then $v = \tilde{v}$.*

COROLLARY 26: *Let v be a continuous, strictly positive weight on a bounded open subset U of $\mathbf{C}^n$ which converges to 0 on the boundary of U. Then the set $\{z : v(z) = \tilde{v}(z)\}$ is non-empty.*

Proof: Since $\mathcal{H}_{v_o}(U)'$ is a dual Banach space it must have extreme point. (In fact, it must have enough extreme points so that their closed convex hull is equal to its unit ball.) ∎

Proposition 24 also allows us to construct examples of continuous strictly decreasing radial weights which are not complete.

We consider the weight $v\colon \Delta \to \mathbf{R}^+$,

$$v(z) = \begin{cases} 1 - |z|, & \text{if } 0 \le |z| \le \frac{1}{4}; \\ (\frac{4}{3}\exp(\sqrt{\log 3(\log|z| + \log 4)}))^{-1}, & \text{if } \frac{1}{4} \le |z| \le \frac{3}{4}; \\ 1 - |z|, & \text{if } \frac{3}{4} \le |z| \le 1. \end{cases}$$

Since the functions $x \to \log x$, $x \to \sqrt{x}$ and $x \to \exp(x)$ are monotone, $v(z)$ is a decreasing function of $|z|$. For $-\log 4 \le t \le \log \frac{3}{4}$,

$$\log(w(e^t)) = \log\left(\frac{1}{v(e^t)}\right) = \log(4/3) + \sqrt{\log 3(t + \log 4)}$$

is not a convex function of t and hence $\log w(z)$ is not a logarithmically convex function of $|z|$. It now follows from the remark before [8, 1.7 Examples] that $w(z) \ne \tilde{w}(z)$ and hence $v(z) \ne \tilde{v}(z)$ for $\frac{1}{4} < |z| < \frac{3}{4}$. This implies that the v-boundary of Δ is different from Δ. By [10, Examples 2.1], $\mathcal{H}_{v_o}(\Delta)''$ is isometrically isomorphic to $\mathcal{H}_v(\Delta)$.

7. Duality, smoothness and rotundness

Our knowledge of the geometry of the unit ball of $\mathcal{H}_{v_o}(U)'$ allows us to deduce a number of results about the Banach space geometry of $\mathcal{H}_{v_o}(U)$ and $\mathcal{H}_v(U)$. In [12] we have seen that it can be used to show that weak and pointwise convergence coincide on bounded sequences in $\mathcal{H}_{v_o}(U)$ and that when v is radial or complete then $\mathcal{H}_{v_o}(U)$ is not isometrically isomorphic to a subspace of c_o. The v-boundary can be used to determine the centraliser of $\mathcal{H}_{v_o}(U)$ and $\mathcal{H}_v(U)$. Here we investigate if $\mathcal{H}_{v_o}(U)$ can be isometrically isomorphic to a dual space and examine the smoothness and rotundness of $\mathcal{H}_{v_o}(U)$ and $\mathcal{H}_v(U)$.

PROPOSITION 27: *Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. Then $\mathcal{H}_{v_o}(U)$ is not isomorphic to a dual space.*

Proof: Suppose that there is a Banach space X whose dual is isometrically isomorphic to $\mathcal{H}_{v_o}(U)$. Then $\mathcal{H}_{v_o}(U)$ is a separable dual space and thus has the Radon–Nikodým property. However, by [11], $\mathcal{H}_{v_o}(U)$ is isomorphic to a subspace of c_o and thus, as c_o is minimal, $\mathcal{H}_{v_o}(U)$ contains a copy of c_o. Since the Radon–Nikodým property is inherited by subspaces, we have a contradiction and therefore $\mathcal{H}_{v_o}(U)$ cannot be isomorphic to a dual space. ∎

THEOREM 28: *Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. Then neither $\mathcal{H}_{v_o}(U)$ nor $\mathcal{H}_v(U)$ is smooth.*

Proof: Since $\mathcal{H}_{v_o}(U)$ is a closed subspace of $\mathcal{H}_v(U)$ and smoothness is a hereditary property, it suffices to prove the result for $\mathcal{H}_{v_o}(U)$. Let us suppose that $\mathcal{H}_{v_o}(U)$ were smooth. Then it would follow from the result of Šmul'yan [29, Proposition 6.9] that every norm-attaining ϕ in the unit sphere of $\mathcal{H}_{v_o}(U)'$ is a weak*-exposed point of the unit ball of $\mathcal{H}_{v_o}(U)'$. Applying the Bishop–Phelps Theorem it now follows that $\mathcal{H}_{v_o}(U)'$ is the norm closure of multiples of its weak*-exposed points. Therefore, given ϕ in $\mathcal{H}_{v_o}(U)'$ can find a sequence of complex numbers $(\lambda_k)_k$ and a sequence $(z_k)_k$ in U such that $\phi = \lim_{k\to\infty} \lambda_k v(z_k)\delta_{z_k}$. Since $\lim_{k\to\infty} |\lambda_k| = \|\phi\|$ the sequence $(\lambda_k)_k$ is bounded. Therefore $(\lambda_k)_k$ has a subsequence $(\lambda_{k_j})_j$ converging to some λ_o in $\mathbf{C}$. As $\overline{U}$ is compact, $(z_{k_j})_j$ has a subsequence $(z_{k_l})_l$ which converges to some point z_o of $\overline{U}$. Hence we have that $\phi = \lim_{l\to\infty} \lambda_{k_l} v(z_{k_l})\delta_{z_{k_l}} = \lambda_o v(z_o)\delta_{z_o}$. Thus every element of $\mathcal{H}_{v_o}(U)'$ has the form $\lambda v(z)\delta_z$ for some λ in $\mathbf{C}$ and some z in $\overline{U}$. As this is impossible, $\mathcal{H}_{v_o}(U)$ and hence $\mathcal{H}_v(U)$ cannot be smooth. ∎

THEOREM 29: *Let U be a balanced bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive radial weight on U which converges to 0 on the boundary of U. Suppose that each point of $\mathcal{B}_v(U)$ is a peak point. Then neither $\mathcal{H}_{v_o}(U)$ nor $\mathcal{H}_v(U)$ is rotund.*

Proof: Consider a non-zero linear functional ϕ on $\mathbf{C}^n$. Then ϕ^2 attains its norm in $\mathcal{H}_{v_o}(U)$ at the points x_o and $-x_o$ for some x_o in $\mathcal{B}_v(U)$. Moreover, by multiplication by a suitable scalar we may assume that $v(x_o)\phi^2(x_o) = v(-x_o)\phi^2(-x_o) = 1$. As x_o is a v-peak point we can find f in $\mathcal{H}_{v_o}(U)$ which peaks at x_o. Then

$$\|\phi^2\|_v = \|f\|_v = \|\frac{1}{2}(\phi^2 + f)\|_v = 1.$$

This shows that each point of the unit sphere of $\mathcal{H}_{v_o}(U)$ cannot be an extreme point. This proves that $\mathcal{H}_{v_o}(U)$ and hence $\mathcal{H}_v(U)$ cannot be rotund. ∎

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